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The symmetric M-matrix and symmetric inverse M-matrix completion problems

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Abstract

The symmetric M-matrix and symmetric M_0-matrix completion problems are solved and results of Johnson and Smith [Linear Algebra Appl. 290 (1999) 193] are extended to solve the symmetric inverse M-matrix completion problem:

(1) A pattern (i.e., a list of positions in an $n \times n$ matrix) has symmetric M-completion (i.e., every partial symmetric M-matrix specifying the pattern can be completed to a symmetric M-matrix) if and only if the principal subpattern R determined by its diagonal is permutation similar to a pattern that is block diagonal with each diagonal block complete, or, in graph theoretic terms, if and only if each component of the graph of R is a clique.

(2) A pattern has symmetric M_0-completion if and only if the pattern is permutation similar to a pattern that is block diagonal with each diagonal block either complete or omitting all diagonal positions, or, in graph theoretic terms, if and only if every principal subpattern corresponding to a component of the graph of the pattern either omits all diagonal positions, or includes *all* positions.

(3) A pattern has symmetric inverse M-completion if and only if its graph is block-clique and no diagonal position is omitted that corresponds to a vertex in a graph-block of order > 2.

The techniques used are also applied to matrix completion problems for other classes of symmetric matrices.

Keywords: Matrix completion; Symmetric M-matrix; Symmetric inverse M-matrix; Partial matrix

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1. Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not. A *completion* of a partial matrix is a matrix obtained by choosing values for the unspecified entries. A *pattern* for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\{1, \dots, n\} \times \{1, \dots, n\}$. A partial matrix *specifies a pattern* if its specified positions are exactly those listed in the pattern. Note that in this paper a pattern does not need to include all diagonal positions.

All matrices and partial matrices discussed here are real. The symbol Π will denote a class of matrices and a *Π-matrix* is a matrix in the class Π. For a particular class Π of matrices, the *Π-matrix completion problem for patterns* asks which patterns have the property that any partial Π-matrix that specifies the pattern can be completed to a Π-matrix. When a pattern has this property, we say it *has Π-completion*.

The answer to the Π-matrix completion problem obviously depends on the definition of partial Π-matrix. For many classes Π of matrices, in order for it to be possible to have a completion of a partial matrix to a Π-matrix, certain obviously necessary conditions must be satisfied. Such obviously necessary conditions are frequently taken as the definition of a partial Π-matrix, [3–6,9,10]. Here we also take this approach of using obviously necessary conditions to define a partial Π-matrix.

In this paper we concern ourselves only with matrix completion problems for patterns. Such problems have been studied for *M*-matrices [4], M_0-matrices [6], inverse *M*-matrices [5,3,9] symmetric inverse *M*-matrices [10], and many other classes. Results on matrix completion problems and techniques are surveyed in [6].

For α a subset of $\{1, \dots, n\}$, the *principal submatrix* $A[\alpha]$ is obtained from the $n \times n$ matrix A by deleting all entries a_{ij} such that $i \notin \alpha$ or $j \notin \alpha$. Similarly, the *principal subpattern* $Q[\alpha] = Q \cap (\alpha \times \alpha)$. The *principal subpattern determined by the diagonal positions* is $Q[\delta]$, where $\delta = \{i \mid (i, i) \in Q\}$.

The *characteristic matrix* of a pattern Q for $n \times n$ matrices is the $n \times n$ matrix C_Q such that $c_{ij} = 1$ if the position (i, j) is in the pattern and $c_{ij} = 0$ if (i, j) is not in the pattern. A pattern Q is *permutation similar* to a pattern R if C_Q is permutation similar to C_R. A pattern is *block diagonal* (for a particular block structure) if its characteristic matrix is block diagonal for that block structure (cf. Section 3 of [6]).

A class Π of matrices is called a *hereditary-sum-permutation-closed* (*HSP*) class if:

(1) every principal submatrix of a Π-matrix is a Π-matrix;
(2) the direct sum of Π-matrices is a Π-matrix;
(3) if A is a Π-matrix and P is a permutation matrix of the same size, then PAP^{-1} is a Π-matrix;
(4) there is a 1×1 Π-matrix.

All of the classes discussed in [6], except those that require entries to be positive (and thus fail condition (2)), are HSP classes. For an HSP class Π we frequently define a partial matrix B to be a partial Π-matrix if any fully specified principal submatrix of

B is a Π-matrix and any sign condition on the entries of a Π-matrix is respected by B (here sign condition includes nonpositive, nonnegative, sign symmetric or weakly sign symmetric, cf. [6]). Using this definition of a partial Π-matrix is referred to as *using the HSP standard definition of a partial Π-matrix*.

A class Σ of matrices is called *symmetric* if every Σ-matrix is symmetric. A partial matrix B is *symmetric* if whenever b_{ij} is specified then so is b_{ji} and $b_{ji} = b_{ij}$. A pattern is *symmetric* (also called "positionally symmetric" and "combinatorially symmetric") if position (i, j) in the pattern implies (j, i) is also in the pattern. A class Σ of matrices is called a *symmetric-hereditary-sum-permutation-closed* (*SHSP*) class if Σ is both symmetric and an HSP class. For an SHSP class Σ, we frequently define a partial Σ-matrix to be a symmetric partial matrix meeting the requirements for a partial matrix of an HSP class. Using this definition of a partial Π-matrix is referred to as *using the SHSP standard definition of partial Π-matrix*.

The matrix A is called *positive stable* (respectively, *semistable*) if all the eigenvalues of A have positive (nonnegative) real part. An *M-matrix* (respectively, *M_0-matrix*) is a positive stable (semistable) matrix with nonpositive off-diagonal entries. There are many equivalent characterizations of M- and M_0-matrices [7]: a matrix with nonpositive off-diagonal entries is an M-matrix (M_0-matrix) if and only if every principal minor is positive (nonnegative). A matrix with nonpositive off-diagonal entries is an M-matrix if and only if it is nonsingular and its inverse is entrywise nonnegative. The notation $M_{(0)}$ will be used to mean "M (respectively, M_0)". The matrix B is an *inverse M-matrix* if B is the inverse of an M-matrix. Equivalently, an inverse M-matrix is a nonsingular, entrywise nonnegative matrix B such that B^{-1} has nonpositive off-diagonal entries. A substantial amount is known about M-matrices, M_0-matrices and inverse M-matrices [7,8,12], including the fact that each of these classes is an HSP class.

Use the HSP standard definition of a partial Π-matrix: a *partial $M_{(0)}$-matrix* is a partial matrix such that any fully specified principal submatrix is an $M_{(0)}$-matrix and all specified off-diagonal entries are nonpositive. A *partial inverse M-matrix* is an entrywise nonnegative partial matrix such that any fully specified principal submatrix is an inverse M-matrix.

We will follow the notation of [10] in referring to a symmetric inverse M-matrix as an SIM matrix and we refer to a symmetric $M_{(0)}$-matrix as an $\mathrm{SM}_{(0)}$-matrix. Note that the three classes SIM, SM and SM_0 are SHSP classes. Use the SHSP standard definition of a partial Π-matrix: a *partial SIM-matrix* is a partial inverse M-matrix that is symmetric. A *partial $SM_{(0)}$-matrix* is a partial $M_{(0)}$-matrix that is symmetric.

With the HSP and SHSP standard definitions of partial Π-matrix, HSP and SHSP classes have two basic properties that are used extensively in the study of matrix completion problems, Lemma 1.1 and Observation 1.2.

Lemma 1.1. *Let Π be an HSP* (*SHSP*) *class using the HSP* (*SHSP*) *standard definition of a partial Π-matrix. If the pattern Q has Π-completion, then so does every principal subpattern $Q[\alpha]$.*

Proof. Let A be a partial Π-matrix specifying $Q[\alpha]$. By (4), there is some 1×1 Π-matrix $[s]$. Extend A to a partial matrix B specifying Q by, for each (i, j) in Q but not in $Q[\alpha]$, setting $b_{ij} = s$ if $i = j$ and 0 otherwise. Then B is a partial Π-matrix because any fully specified principal submatrix $B[\beta]$ of B is permutation similar to $A\ [\alpha \cap \beta] \oplus [s] \oplus \cdots \oplus$ [s], which is a Π-matrix by (2). So $B[\beta]$ is a Π-matrix by (3). By hypothesis, B can be completed to a Π-matrix C. Then $C[\alpha]$, which completes A, is a Π-matrix by (1). □

Our graph terminology follows [6]. For a symmetric pattern Q, the *pattern-graph* of Q is the graph having $\{1, \ldots, n\}$ as its vertex set and, as its set of edges, the set of (unordered) pairs $\{i, j\}$ such that position (i, j) (and therefore also (j, i)) is in Q. If G is the pattern-graph of Q, then the pattern-graph of a principal subpattern $Q[\alpha]$ is $\langle\alpha\rangle$, the subgraph induced by α. The principal subpattern $Q[\alpha]$ and the induced subgraph $\langle\alpha\rangle$ are said to *correspond*; in particular, the vertex v and diagonal position (v, v) correspond. Renaming the vertices of a pattern-graph is equivalent to applying a permutation similarity to the pattern.

A *component* of a graph is a maximal connected subgraph. A *cut-vertex* of a connected graph is a vertex whose deletion disconnects the graph; more generally, a cut-vertex is a vertex whose deletion disconnects the component containing it. A graph is *nonseparable* if it is connected and has no cut-vertices. A *block* of a graph is a subgraph that is nonseparable and is maximal with respect to this property. A (sub)graph is called a *clique* if it contains all possible edges between its vertices. A graph is *block-clique* if every block is a clique. Block-clique graphs are called "1 chordal" in [10].

For matrices and patterns that need not be symmetric, digraphs must be used. Let A be a (fully specified) $n \times n$ matrix. The *nonzero-digraph* of A is the digraph having as vertex set $\{1, \ldots, n\}$, and, as its set of arcs, the set of ordered pairs (i, j) such that both i and j are vertices with $i \neq j$ and $a_{ij} \neq 0$. For a pattern Q that need not be symmetric, the *pattern-digraph* of Q is the digraph having $\{1, \ldots, n\}$ as its vertex set and members (i, j) of Q with $i \neq j$ as its arcs. A digraph is *transitive* if the existence of a path from v to w implies the arc (v, w) is in the digraph. Recall that the nonzero-digraph of any inverse M-matrix is transitive [12].

Observation 1.2. *Let Π be an HSP* (*SHSP*) *class. If the pattern Q is permutation similar to a block diagonal pattern in which each diagonal block has Π-completion, then Q has Π-completion by* (2) *and* (3). *Equivalently, if each principal subpattern of Q corresponding to a component of a pattern-digraph* (*pattern-graph*) *has Π-completion, then the pattern has Π-completion.*

The results in Lemma 1.1 and Observation 1.2 are already known for $\mathrm{SM}_{(0)}$-matrices and SIM-matrices [6].

Johnson and Smith [10] determined that a symmetric pattern that includes all diagonal positions has SIM completion if and only if its pattern-graph is block-clique.

More general patterns that may omit some diagonal positions are classified as to SIM completion in the following section. All patterns are classified as to $\mathrm{SM}_{(0)}$-completion in Section 3.

2. Determination of patterns having SIM completion

It is well known that a graph G is block-clique if and only if for every cycle $v_1, v_2 \ldots, v_k, v_1$ of G, the induced subgraph $\langle\{v_1, v_2 \ldots, v_k\}\rangle$ is a clique [9].

Theorem 2.1. *Let Q be a symmetric pattern and let G be its pattern-graph. If Q has SIM completion, then G is block-clique and the diagonal positions corresponding to the vertices of every cycle in G are all included in Q.*

Proof. Suppose Q and G do not have the required property. Then G is contains a cycle whose induced subgraph is not a clique or Q omits the diagonal position corresponding to a vertex in a cycle. Let Γ be a shortest troublesome cycle. By renaming vertices if necessary, assume $\Gamma = 1, 2, \ldots, k, 1$ with $k > 2$, and $\langle\{1, \ldots, k\}\rangle$ is not a clique or diagonal position (k, k) is not in Q.

Suppose first that $\langle\{1, \ldots, k\}\rangle$ does not contain any chord of Γ. If $k > 3$, then $Q[\{1, \ldots, k\}]$ does not contain any complete principal subpattern of size larger than 2×2, because all chords are omitted. When $k = 3$, $(3, 3)$ must be omitted from Q. Thus, in either case, $Q[\{1, \ldots, k\}]$ does not contain any complete principal subpattern of size larger than 2×2. Define a $k \times k$ partial matrix B specifying $Q[\{1, \ldots, k\}]$ by setting $b_{ii} = 2$ for $(i, i) \in Q[\{1, \ldots, k\}]$, setting $b_{ii+1} = 1 = b_{i+1i}$ for $i = 1, \ldots, k-1$, and setting all other specified entries (including b_{1k} and b_{k1}) equal to 0. Since the only completely specified principal submatrices are 2×2 or smaller, and these are SIM matrices, B is a partial SIM matrix. But B cannot be completed to a SIM matrix because the nonzero-digraph of any completion of B is not transitive, since $b_{12} = \cdots = b_{k-1k} = 1$ and $b_{1k} = 0$. So $Q[\{1, \ldots, k\}]$ does not have SIM completion.

Now suppose $k > 3$ and $\langle\{1, \ldots, k\}\rangle$ contains a chord of Γ. Each of the two pieces of Γ on either side of the chord, together with the chord, forms a shorter cycle. By the minimal length assumption, Q must include all diagonal positions corresponding to vertices in these two shorter cycles. Hence, Q includes all diagonal positions $(1, 1), \ldots, (k, k)$. Then $\langle\{1, \ldots, k\}\rangle$ is not a clique, and thus is not block-clique. So $Q[\{1, \ldots, k\}]$ is a symmetric pattern that includes all diagonal positions and whose graph is not block-clique, and therefore does not have SIM completion [10].

In either case Q does not have SIM completion because $Q[\{1, \ldots, k\}]$ does not. □

The properties that for any vertex v that appears in a cycle of G, $(v, v) \in Q$, and the induced subgraph of a cycle of G must be a clique are the graph analog of the pattern-digraph condition “path-clique”, which is necessary for a (not necessarily symmetric) pattern to have inverse M completion [5]. (In a digraph D, an *alternate path to a single arc* is a path $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ with $k > 2$, such that (v_1, v_k) is an arc of D. A pattern-digraph is called *path-clique* if the induced subdigraph of any alternate path to a single arc is a clique and the diagonal position (v_i, v_i) is in the pattern for every vertex v_i in the path).

Theorem 2.2. *A symmetric pattern Q has SIM completion if and only if its pattern-graph G is block-clique and the diagonal position (v, v) is in Q for every vertex v in a block of order > 2.*

Proof. (only if) By Theorem 2.1, G is block-clique. Let v be a vertex in a block H of order > 2. There are two other distinct vertices u and w in H. Since H is a clique, $\{v, u\}$, $\{u, w\}$ and $\{w, v\}$ are in H, and v occurs in a cycle. So by Theorem 2.1 (v, v) is in Q.

(if) Let H be a block of G of order > 2. By hypothesis, H is a clique and (v, v) is in Q for every vertex v in H. Thus the principal subpattern $Q[\eta]$ corresponding to H contains *all* positions, so $Q[\eta]$ trivially has SIM-completion.

Any symmetric pattern for 2×2 matrices has SIM completion [6, remark following Lemma 4.8], so the principal subpattern of Q corresponding to any block of G of order 2 has SIM completion. Thus the principal subpattern of Q corresponding to each block of G is has SIM completion, and so Q has SIM completion by Corollary 5.6 of [6]. □

The following example exhibits patterns that do not include all diagonal positions, one having SIM completion and other not having SIM completion. In the diagrams of the pattern-graphs, if a diagonal position (v, v) is in the pattern, then vertex v is indicated by a solid black dot (–•–); if (v, v) is omitted, then vertex v is indicated by a hollow circle (–○–) (this follows the notation of [5,6]).

Example 2.3. The pattern $Q_1 = \{$(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,3), (4,4), (4,5), (4,6), (4,7), (5,3), (5,4), (5,5), (5,6), (5,8), (6,3), (6,4), (6,5), (6,6), (7,4), (8,5), (8,9), (8,11), (9,8), (9,10), (10,9), (10,10), (11,8), (11,11), (11,12), (11,13), (12,11), (12,12), (12,13), (13,11), (13,12), (13,13)$\}$, whose pattern-graph is shown in Fig. 1(a), has SIM completion. The pattern Q_2 obtained from Q_1 by deleting the diagonal position (3,3), whose pattern-graph is shown in Fig. 1(b), does not.

3. Determination of patterns having $\mathrm{SM}_{(0)}$-completion

A partial $M_{(0)}$-matrix with all diagonal entries specified can be completed to an $M_{(0)}$-matrix if only if its zero completion (i.e., the result of setting all unspecified entries to 0) is an $M_{(0)}$-matrix, cf. [6,9]. Since a partial $\mathrm{SM}_{(0)}$-matrix is an $M_{(0)}$-

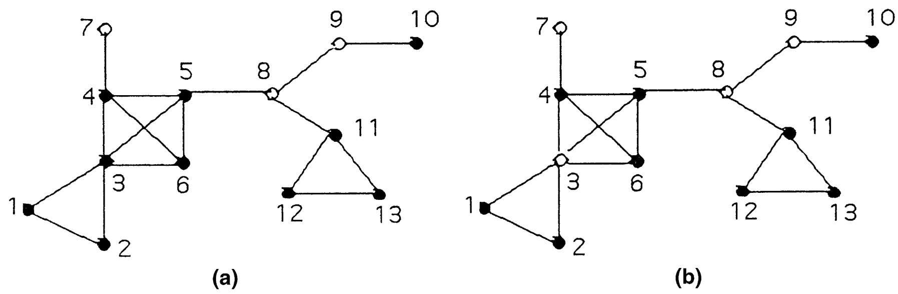

Fig. 1. (a) Q_1 has SIM completion; (b) Q_2 does not have SIM completion.

matrix, and the zero completion of a partial $\mathrm{SM}_{(0)}$-matrix is symmetric, a partial $\mathrm{SM}_{(0)}$-matrix with all diagonal entries specified can be completed to an $\mathrm{SM}_{(0)}$-matrix if only if its zero completion is an $\mathrm{SM}_{(0)}$-matrix.

Lemma 3.1. *If a symmetric pattern Q has $SM_{(0)}$-completion and includes positions (a, a), (b, b), (c, c), (a, b), (b, a), (b, c), and (c, b) with $a < b < c$, then Q also includes (a, c) and (c, a).*

Proof. Suppose Q does not include (a, c) and (c, a). Then the partial matrix SM-matrix

$$A = \begin{bmatrix} 4 & -3 & ? \\ -3 & 4 & -3 \\ ? & -3 & 4 \end{bmatrix}$$

specifies $Q[\{a,b,c\}]$and cannot be completed to an SM_0-matrix because the zero-completion of A has determinant -8. Thus Q does not have $\mathrm{SM}_{(0)}$-completion. □

Theorem 3.2. *Let Q be a symmetric pattern with the property that if (v, w) is in Q, then (v, v) and (w, w) are both in Q. Then the following are equivalent*:

(1) *Q has $SM_{(0)}$-completion.*
(2) *Q is permutation similar to a block diagonal pattern with each diagonal block containing all positions or consisting of a single omitted diagonal position.*
(3) *Each component of its pattern-graph is a clique.*

Proof. The equivalence of the (2) and (3) is immediate from the hypothesis about Q. If Q satisfies (3), then Q has $\mathrm{SM}_{(0)}$-completion by Observation 1.2.

Let Q have $\mathrm{SM}_{(0)}$-completion. Let u and v be vertices of H, a component of order >1 of the pattern-graph of Q. Since (w, w) is in Q for every vertex visited by a path in H connecting u and v, we may apply Lemma 3.1 to eliminate one at a time from that path any vertex other than u and v. Hence $\{v, u\}$ is in H and H is a clique. □

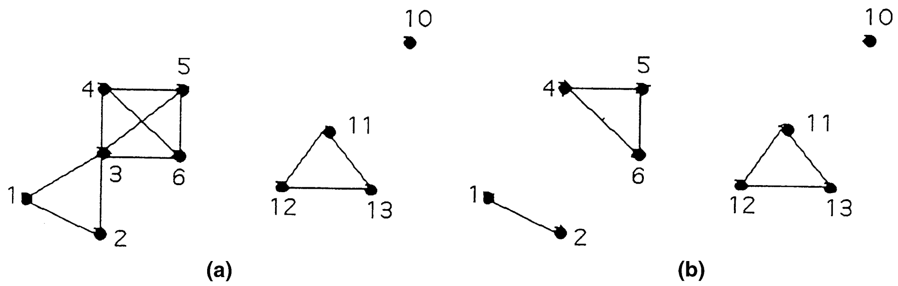

Fig. 2. (a) $Q_1[\delta]$; (b) $Q_2[\delta]$.

A symmetric pattern Q has SM-completion if and only if $Q[\delta]$ does [6, Theorem 4.6]. The principal subpattern $Q[\delta]$ satisfies the hypotheses of Theorem 3.2, so this completes the determination of patterns having SM-completion.

Corollary 3.3. *A symmetric pattern Q has SM-completion if and only if $Q[\delta]$ is permutation similar to a pattern that is block diagonal with all positions in each of the blocks on the diagonal in the pattern; in graph theoretic language, if and only if each component of the pattern-graph of $Q[\delta]$ is a clique.*

Example 3.4. The pattern Q_1 in Example 2.3 does not have SM-completion because one of the components of the pattern-graph of $Q_1[\delta]$ is not a clique. The pattern Q_2 in Example 2.3 has SM-completion, because each component of $Q_2[\delta]$ is a clique. The pattern-graphs of $Q_1[\delta]$ and $Q_2[\delta]$ are shown in Figs. 2(a) and (b), respectively.

Example 3.5. The partial SM_0-matrix

$$A = \begin{bmatrix} 0 & -1 \\ -1 & ? \end{bmatrix}$$

cannot be completed to an SM_0-matrix because the determinant of any completion of A equals -1.

Thus, neither Q_1 nor Q_2 from Example 2.3 has SM_0-completion, because both contain the principal subpattern $R = \{(4,4), (4,7), (7,4)\}$.

Lemma 3.6. *Let Π be an HSP (SHSP) class (using the HSP (SHSP) standard definition of a partial Π-matrix) such that $\{(a,a),\ (a,b),\ (b,a)\}$ (with $a \neq b$) does not have Π-completion. Then if a symmetric pattern Q has Π-completion, Q is permutation similar to a block diagonal pattern in which every diagonal block either includes all diagonal positions or omits all diagonal positions, i.e., in graph theoretic terms,* *every principal subpattern R corresponding to a component H of the pattern-graph G of Q includes all diagonal positions or omits all diagonal positions.*

Proof. Suppose R includes (v, v) and omits (w, w). Since H is a component, it is connected, and it contains a path $\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{k-1}, u_k\}$ from vertex $v = u_1$ to vertex $w = u_k$. Let t be the number such that R includes $(u_1, u_1), \ldots, (u_t, u_t)$ and R does not include (u_{t+1}, u_{t+1}). Then $Q[\{u_t, u_{t+1}\}] = \{(u_t, u_t), (u_t, u_{t+1}), (u_{t+1}, u_t)\}$. Thus by the hypothesis, $Q[\{u_t, u_{t+1}\}]$ does not have Π-completion, and so neither does Q, by Lemma 1.1. □

Theorem 3.7. *Let Q be a symmetric pattern and let G be its pattern-graph. Then Q has* SM_0*-completion if and only if Q is permutation similar to a pattern that is block diagonal in which each diagonal block either omits all diagonal positions or includes* all *positions, i.e., in graph theoretic language, if and only if every principal subpattern corresponding to a component of G either omits all diagonal positions, or includes* all *positions.*

Proof. If Q has SM_0-completion, Lemma 3.6 shows that after permutation similarity the diagonal blocks have either no or all diagonal positions. Theorem 3.2 shows that in the latter case the block includes all positions.

Conversely, a pattern that omits all diagonal positions has SM_0-completion [6, Theorem 4.7]. A pattern that includes *all* positions trivially has SM_0-completion. Since Q is permutation similar to a block-diagonal pattern in which each diagonal block has SM_0-completion, Q has SM_0-completion by 1.2. □

Corollary 3.8. *Any symmetric pattern that has* SM_0*-completion also has SM-completion* (*cf.* 3.3), *but the converse is false* (*cf.* 3.5). *A symmetric pattern that includes all diagonal positions and has SM-completion also has* SM_0*-completion* (*cf.* 3.2).

The next example shows that the assumption in Lemma 3.6 that the pattern is symmetric is necessary. A matrix is a P_0-*matrix* if every principal minor is nonnegative. Use the HSP standard definition of a partial Π-matrix: a *partial* P_0-*matrix* is a partial matrix such that any fully specified principal submatrix is an P_0-matrix. The pattern $\{(a, a),\ (a, b),\ (b, a)\}$ (with $a \neq b$) does not have P_0-completion (cf. Example 3.5), so Lemma 3.6 applies to the class of P_0-matrices.

Example 3.9. The pattern {(1,1), (1,2), (2,2), (2,3), (3,1)} (whose pattern-digraph is a 3-cycle) has P_0-completion, because it is asymmetric [1], and neither contains nor omits all diagonal positions.

We can also use Lemma 3.6 to complete the classification of patterns for other classes of symmetric matrices. The matrix A is *doubly nonnegative* (*DN*) if A is entrywise nonnegative and positive semidefinite. The matrix A is *completely positive* (*CP*) if A is entrywise nonnegative and $A = BB^{\mathrm{T}}$ for some entrywise nonnegative $n \times m$ matrix B (the requirement that A be entrywise nonnegative is clearly redundant, but helps clarify the interpretation of using the SHSP standard definition of a partial CP-matrix). The classes DN and CP are SHSP classes [2]. Use the SHSP standard definitions of a partial Π-matrix: a *partial DN-matrix* is an entrywise nonnegative symmetric partial matrix such that any fully specified principal submatrix is a DN-matrix. A *partial CP-matrix* is an entrywise nonnegative symmetric partial matrix such that any fully specified principal submatrix is a CP-matrix. Drew and Johnson [2] established that a symmetric pattern that includes the diagonal has DN-(CP-)completion if and only if its pattern-graph is block-clique. The partial matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$$

shows that $\{(a, a), (a, b), (b, a)\}$ does not have DN- or CP-completion. Since any diagonally dominant nonnegative symmetric matrix is CP [11] (and thus DN), a pattern that omits all diagonal positions has CP- and DN-completion.

Corollary 3.10. *Let Q be a symmetric pattern and let G be its pattern-graph. Then Q has DN-(CP-)completion if and only if every principal subpattern corresponding to a component H of G either omits all diagonal positions, or includes all diagonal positions and H is block-clique.*

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